$\mathrm{Out}(F_n)$ **AND THE SPECTRAL GAP CONJECTURE**

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ABSTRACT. For $n > 2$, given $\phi_1, \ldots, \phi_n$ randomly chosen isometries of S^2, it is well-known that the group Γ generated by $\phi_1, \ldots, \phi_n$ acts ergodically on S^2. It is conjectured in [GJS] that for almost every choice of $\phi_1, \ldots, \phi_n$ this action is *strongly ergodic*. This is equivalent to the spectrum of $\phi_1 + \phi_1^{-1} + \cdots + \phi_n + \phi_n^{-1}$ as an operator on $L^2(S^2)$ having a spectral gap, i.e. all eigenvalues but the largest one being bounded above by some $\lambda_1 < 2n$. (The largest eigenvalue λ_0, corresponding to constant functions, is $2n$.)

In this article we show that if $n > 2$, then either the conjecture is true or almost every n-tuple fails to have a gap. In fact, the same result is holds for any n-tuple $\phi_1, \ldots, \phi_n$ in any any compact group K that is an almost direct product of $SU(2)$ factors with $L^2(S^2)$ replaced by $L^2(X)$ where X is any homogeneous K space. A weaker result is proven for $n = 2$ and some conditional results for similar actions of F_n on homogeneous spaces for more general compact groups.

1. INTRODUCTION.

Let $\phi_1, \ldots, \phi_n$ be any finite collection of elements of $SU(2)$ and let $L_0^2(SU(2))$ be the orthogonal complement of the constant functions in $L^2(SU(2))$. The operator $\phi_1 + \phi_1^{-1} + \cdots + \phi_n + \phi_n^{-1}$ is a self-adjoint operator on $L_0^2(SU(2))$ and has discrete spectrum which is a subset of $\mathbb{R}^+$. Let λ_1 be the supremum of the eigenvalues for this operator. It is clear that $\lambda_1 \leq 2n$. The following is conjectured in [GJS]:

Conjecture 1.1. *For $n \geq 2$ and almost every collection $\phi_1, \ldots, \phi_n$, we have $\lambda_1 < 2n$ for $\phi_1 + \phi_1^{-1} + \cdots + \phi_n + \phi_n^{-1}$.*

This conjecture is referred to as the spectral gap conjecture and is a question in [LPS]. The conjecture is only known for n-tuples which have, up to conjugacy in $SU(2)$, all matrix entries of all ϕ_i algebraic. This is a recent result of Bourgain and Gamburd building on earlier work of Gamburd, Jakobson and Sarnak [BG, GJS]. This set of n-tuples for which the conjecture is known has zero measure. See [F, Theorem 3.2] and [KR] and the references there for weaker related results.

The main result of this paper is the following:

Theorem 1.2. *Assume $n \geq 3$. Then either $\lambda_1 < 2n$ for almost every $\phi_1, \ldots, \phi_n$ or $\lambda_1 = 2n$ for almost every $\phi_1, \ldots, \phi_n$.*

In particular, by Theorem 1.2, to prove Conjecture 1.1 it suffices to establish a spectral gap for any set of positive measure in $SU(2)^n$.

It is well-known that for almost every $\phi_1, \ldots, \phi_n$, the group generated by $\phi_1, \ldots, \phi_n$ is a free group on n generators, F_n. The space of n-tuples $\phi_1, \ldots, \phi_n$ can be parametrized as $\mathrm{Hom}(F_n, SU(2))$. The main new ingredient in Theorem 1.2 is the use of symmetries of F_n and in particular ergodicity of a the action of $\mathrm{Aut}(F_n)$ on $\mathrm{Hom}(F_n, SU(2))$ where $\mathrm{Aut}(F_n)$ is the automorphism group of F_n.

Theorem 1.2 remains true when $SU(2)$ is replaced by any compact Lie group K which is an almost direct product of copies of $SU(2)$ and $SU(1)$. For any compact Lie group with $SU(1) = S^1$ factors, analogues of Theorem 1.2 are not interesting, as it is easy to see in that case that there is no spectral gap on a set of full measure.

The interest in spectral gaps for finite collections of elements in $SU(2)$ originally derives from the Banach-Ruscewiecz conjecture and its proof. This states that, for $m > 1$, the unique finitely additive rotationally invariant measure on S^m is the Haar measure. Rosenblatt showed that this was equivalent to finding a finite subset $\phi_1, \ldots, \phi_n$ in $\mathrm{Isom}(S^m)$ with a spectral gap for the action on $L_0^2(S^m)$ [Ro]. For $n > 1$, a spectral gap for $\phi_1, \ldots \phi_n$ in $\mathrm{Isom}(S^m)$ on $L_0^2(S^m)$ is easily seen to be equivalent to a spectral gap for $\phi_1, \ldots \phi_n$ on $L_0^2(\mathrm{Isom}(S^m))$ since the same representations of $\mathrm{Isom}(S^m)$ occur in $L_0^2(S^m)$ and $L_0^2(\mathrm{Isom}(S^m))$, just with different multiplicities. Similarly, Conjecture 1.1 is equivalent to the same conjecture with $SO(3)$ in place of $SU(2)$. For $n > 3$ Sullivan and Margulis independently exhibited such subsets with a spectral gap, each by finding a homomorphism from a group Γ with property (T) of Kazhdan to $\mathrm{Isom}(S^m)$ [Ma, Su]. For $n = 2, 3$, subsets of $\mathrm{Isom}(S^m)$ with a spectral gap were first exhibited by Drinfeld using methods of automorphic forms [Dr]. Later work on the subject was motivated by the fact that if $\phi_1, \ldots, \phi_n$ have a spectral gap, then the orbits under the resulting action of F_n on S^m equidistribute with exponential speed. In [LPS], the authors show how to find $\phi_1, \ldots \phi_n$ with optimal equidistribution properties, again using deep results on automorphic forms. In [GJS], the authors prove the existence of $\phi_1, \ldots \phi_n$ in $SU(2)$ with a spectral gap without using heavy machinery from the theory of automorphic forms and also discuss several related issues. For more discussion see [GJS, Lu, Sa].

As mentioned above, the key step in the proof of all results here is to use the ergodic theory of the action of $\mathrm{Aut}(F_n)$ on $\mathrm{Hom}(F_n, K)$. In fact, since it is easy to check that the spectral gap is invariant under conjugation in $SU(2)$ it is easier to work with the action of $\mathrm{Out}(F_n)$ on $\mathrm{Hom}(F_n, SU(2))/SU(2)$ instead. The group $\mathrm{Aut}(F_n)$ of automorphisms of F_n acts on $\mathrm{Hom}(F_n, K)$ and this action descends to an action of the outer automorphism group $\mathrm{Out}(F_n)$ on $\mathrm{Hom}(F_n, K)/K$. The $\mathrm{Aut}(F_n)$ action preserves the measure on $Hom(F_n, K)$ given by identifying this space with K^n and taking Haar measure. The $\mathrm{Out}(F_n)$ action preserves the measure on $\mathrm{Hom}(F_n, K)/K$ given by

realizing $\mathrm{Hom}(F_n, K)$ as K^n, taking Haar measure on each factor, and dividing by the conjugation action of K to obtain the quotient $\mathrm{Hom}(F_n, K)/K$. The dynamics of this action have received relatively little attention, but the analogous action of the mapping class group on $\mathrm{Hom}(S, K)/K$ where S is the fundamental group of a surface, has been studied more extensively, see the recent survey [Go3] which is also a good introduction to dynamics of group actions on representation varieties. Essentially the only known result for the action of $\mathrm{Out}(F_n)$ on $\mathrm{Hom}(F_n, K)/K$ is due to Goldman who shows that the action is weakly mixing when $k \geq 3$ and K is an almost direct product of $SU(2)$ and $SU(1)$ factors. This is proven in [Go2] using the main results of [Go1]. The remaining ingredient in the proof of Theorem 1.2 is to construct a measurable function f on $\mathrm{Hom}(F_n, SU(2))/SU(2)$ that is $\mathrm{Out}(F_n)$ invariant and takes the value 1 for actions with a spectral gap and the value zero for actions without a spectral gap. We will construct the function f in §3. In the next section, we state some other variants of Theorem 1.2. and recall some results about the ergodic theory of actions on moduli spaces from [Go1, Go2, PX1, PX2]. In section §3 we prove all of our results.

2. FURTHER RESULTS AND GROUP ACTIONS ON REPRESENTATION VARIETIES.

A key ingredient in our proof of Theorem 1.2 is the following result of Goldman.

Theorem 2.1 (Goldman). *Let K be a compact group which is an almost direct product of $SU(2)$ and $SU(1)$ factors. If $n > 2$, then the action of $\mathrm{Out}(F_n)$ on $\mathrm{Hom}(F_n, K)/K$ is ergodic.*

When $n = 2$, there are non-constant function on $\mathrm{Hom}(F_2, SU(2))/SU(2)$ or even $\mathrm{Hom}(F_n, K)/K$ which are easily seen to be $\mathrm{Out}(F_2) = SL(2, \mathbb{Z})$ invariant. For $K = SU(2)$, one such function is simply $g(\rho) = \mathrm{Trace}([\rho(a), \rho(b)])$ where a, b are a basis for F_2. In this case invariance follows from the fact that the set of commutators is $\mathrm{Aut}(F_2)$ invariant and that trace is conjugation invariant. For general K, we need a few facts before we can define an analogous function. It is well-known that every element of K is contained in a maximal torus $T < K$ and that all such tori are conjugate in K. This allows us to parametrize the conjugacy classes in K as T/W where $W < K$ is the Weyl group, i.e. the normalizer of T divided by the centralizer of T. Given K, we define $g : \mathrm{Hom}(F_2, K)/K \to T/W$ by taking the representative of the conjugacy class of $[\rho(a), \rho(b)]$. Again this function is invariant, since the commutator is $\mathrm{Out}(F_2)$ invariant. The following result of Pickrell and Xia is essentially [PX1, Theorem 2.1.4]. In the case where K is as in Theorem 2.1, the result is contained in [Go1].

Theorem 2.2 (Pickrell-Xia). *For any compact Lie group K, the map $g : \mathrm{Hom}(F_2, K)/K \to T/W$ defined above is an ergodic decomposition for the action of $\mathrm{Out}(F_2)$ on $\mathrm{Hom}(F_2, K)/K$.*

In both [Go1] and [PX1], $\mathrm{Out}(F_2)$ is considered as the mapping class group of a once punctured torus.

By viewing g as an ergodic decomposition, we are writing the measure on $\mathrm{Hom}(F_2, K)/K$ as an integral over T/W of measures on the level sets of g. In fact, level sets of g are generically smooth submanifolds and these measures are smooth measures. It is clear that we can view g as a function on $\mathrm{Hom}(F_2, K)$ instead. In the following result, λ_1 is again the supremum of eigenvalues for the operator $\rho(a) + \rho(b) + \rho(a)^{-1} + \rho(b)^{-1}$ on $L_0^2(K)$.

Theorem 2.3. *Let a, b be a basis for F_2 and let K be any compact group. Let X be the subset of $\mathrm{Hom}(F_2, K)$ such that $\lambda_1 < 4$ for $\rho(a) + \rho(b) + \rho(a)^{-1} + \rho(b)^{-1}$. Then for almost every a in the image of g (with the push-forward measure), the set $X \cap g^{-1}(a)$ has either zero measure or full measure in $g^{-1}(a)$.*

This theorem is proven exactly as Theorem 1.2, using Theorem 2.2 in place of Theorem 2.1. The main obstruction to a variant of Theorem 1.2 for general K is the lack of an analogue of the main result of [Go2] for general K. The following conditional result also follows from the proof of Theorem 1.2.

Theorem 2.4. *Let $a_1, a_2, \ldots, a_n$ be a basis for F_n. Let X be the subset of $\mathrm{Hom}(F_n, K))$ such that $\lambda_1 < 2n$ for $\rho(a_1) + \rho(a_1{}^{-1}) + \cdots + \rho(a_n) + \rho(a_n)^{-1}$. Then if $n > 2$, the measure of X is either 0 or 1 provided the $\mathrm{Out}(F_n)$ action on $\mathrm{Hom}(F_n, K)/K$ is ergodic.*

3. PROOFS OF THE MAIN RESULTS.

The proof of all results here depend on another, equivalent, definition of the spectral gap. The following discussion and the first two lemmas of this section are standard, but we include them for completeness. We can define the *spectral gap* for a unitary representation ρ of a finitely generated group Γ with generating set S on a Hilbert space $\mathcal{H}$ to be the largest ε such that for each $v \in \mathcal{H}$ there is some γ in S such that:

$$\|v - \rho(\gamma)v\| \geq \varepsilon \|v\|.$$

Note that the spectral gap depends on the generating set S. This is because a choice of generating sets determines a particular basis of neighborhoods of the trivial representation in the Fell topology. That having a non-zero spectral gap in this sense is equivalent to the definition of spectral gap given above is more or less immediate from the definition of the Fell topology on the unitary dual of group, but we give a proof below for completeness. The following standard lemma shows that having a non-zero spectral gap is independent of generating set.

Lemma 3.1. *Let Γ be a finitely generated group and let S_1 and S_2 be two generating sets for Γ. Then Γ has non-zero spectral gap for S_1 if and only if it has a non-zero spectral gap for S_2.*

Proof. Let ε be the spectral gap for (Γ, S_1). Let n be the smallest integer such that every element of S_1 can be written as a word of length n in the generators S_2. Then we claim that the spectral gap for (Γ, S_2) is at least $\frac{\varepsilon}{n}$. If not we have a vector v in a representation σ of Γ on a Hilbert space $\mathcal{H}$ such that

$$\|\sigma(\gamma)v - v\| < \frac{\varepsilon}{n}$$

for all γ in S_2. Writing any $\tilde{\gamma} \in S_1$ as $\gamma_1 \cdots \gamma_i$ where $i \leq n$ and using a standard telescoping sum argument, this implies that

$$\|\sigma(\tilde{\gamma})v - v\| < \varepsilon$$

a contradiction. Reversing the roles of S_1 and S_2 completes the proof. $\square$

The following standard lemma implies that our two definitions of spectral gap are equivalent.

Lemma 3.2. *Let Γ be a finitely generated group with generators $g_1, g_2, \ldots, g_n$ and ρ a unitary representation of Γ on a Hilbert space $\mathcal{H}$. Then ρ has a spectral gap if and only if the norm the operator $\rho(g_1) + \cdots + \rho(g_n)^{-1}$ is strictly less than $2n$.*

Proof. If we adjoin the identity to any generating set as γ_{n+1}, the norm of the operator $\rho(g_1) + \cdots + \rho(g_{n+1})^{-1}$ is simply 2 plus the norm of the operator $\rho(g_1) + \cdots + \rho(g_n)^{-1}$. Combined with Lemma 3.1 this means it suffices to prove the current lemma for generating sets that contain the identity. This reduces to the following elementary fact about Hilbert spaces: given k unit vectors $v_1, \ldots, v_k$ not all of which are equal, $\|\frac{1}{k}\sum v_k\| < 1 - f(v_1, \ldots, v_k)$ where f is a positive function of the diameter of the set $v_1, \ldots, v_k$ which goes to zero only when the diameter goes to zero. (It is not too hard to write down f explicitly.) $\square$

We now prove a lemma that suffices to prove all the theorems in the previous section.

Lemma 3.3. *Fix a finitely generated group Γ and a generating set $\gamma_1, \ldots, \gamma_n$. Define the function $\mathrm{gap}(\rho)$ to be the spectral gap of $(\rho(\Gamma), S)$ acting on $L_0^2(K)$ where K is compact Lie group and ρ is in $\mathrm{Hom}(\Gamma, K)$. Then gap is a well-defined measurable function on $\mathrm{Hom}(\Gamma, K)/K$.*

Proof. Fix a Riemannian metric on K invariant under both left and right multiplication. Let Δ be the associated Laplacian. Recall that $L_0^2(K)$ decomposes as a Hilbertian direct sum

$$\oplus_\lambda V_\lambda$$

where λ runs over non-zero eigenvalues of Δ and each V_λ is a bi-K invariant finite dimensional space of smooth functions on K. Let $S(V_\lambda)$ be the unit sphere in V_λ. The action of K on $S(V_\lambda)$ is smooth, so for every representation $\rho : \Gamma \rightarrow K$, we have a smooth action ρ_λ of Γ on $S(V_\lambda)$ and ρ_λ depend smoothly on ρ. Therefore the function $\rho(\gamma_i)v - v$ is a smooth

function on $\mathrm{Hom}(\Gamma, K) \times S(V_\lambda)$. The function $\|\rho(\gamma_i)v - v\|$ is continuous on $\mathrm{Hom}(\Gamma, K) \times S(V_\lambda)$ and so the function $\mathrm{gap}_\lambda(\rho, \gamma_i) = \min_{S(V_\lambda)} \|\rho(\gamma_i)v - v\|$ is continuous on $\mathrm{Hom}(\Gamma, K)$. Therefore:

$$\widetilde{\mathrm{gap}}(\rho) = \inf_\lambda \max_{\gamma_1, \ldots, \gamma_n} \mathrm{gap}_\lambda(\rho, \gamma_i)$$

is a measurable function on $\mathrm{Hom}(\Gamma, K)$. It is immediate from the definition that $\widetilde{\mathrm{gap}}(\rho)$ is in fact the spectral gap for (ρ, S) and that it is invariant under conjugation. $\square$

We now proceed to prove the theorems stated in the introduction.

Proof of Theorems 1.2, 2.3 and 2.4. We define a function $\mathrm{pgap}(\rho)$ on the space $\mathrm{Hom}(\Gamma, K)/K$ such that $\mathrm{pgap}(\rho) = 1$ if $\mathrm{gap}(\rho) > 0$ and $\mathrm{pgap}(\rho) = 0$ otherwise. By Lemma 3.1 the function pgap is $\mathrm{Out}(F_n)$ invariant, an automorphism of F_n simply changes the generating set for which we want a gap. By Lemma 3.3 the function pgap is measurable.

To complete the proof of Theorem 1.2, we note that by Theorem 2.1, the action of $\mathrm{Out}(F_n)$ on $\mathrm{Hom}(\Gamma, K)/K$ is ergodic as long as K is locally a product of $SU(2)$ and $SU(1)$ factors and $n > 2$. This implies that pgap is either almost everywhere one or almost everywhere zero. Similarly, to complete the proof of Theorem 2.3, we recall that by Theorem 2.2, the level sets of the function g defined in §2 are ergodic components for the action of $\mathrm{Out}(F_2)$ on $\mathrm{Hom}(F_2, K)/K$. This immediately implies the statement of the theorem. The proof of Theorem 2.4 is the same as the proof of Theorem 1.2. $\square$

4. SPECULATION AND QUESTIONS

Theorem 1.2 allows one to prove Conjecture 1.1 by proving the existence of a spectral gap on any set of positive measure in $\mathrm{Hom}(F_n, SU(2))$. It also leaves one with the impression that the large group of symmetries of F_n might be relevant to a proof of Conjecture 1.1.

The following conjecture seems natural in the context of this work:

Conjecture 4.1. *The representation of* $\mathrm{Out}(F_n)$ *on* $L_0^2(\mathrm{Hom}(F_n, K)/K)$ *has a spectral gap for* $n > 3$.

One can reformulate this as saying that the trivial representation is isolated in the representation of $\mathrm{Out}(F_n)$ on $L^2(\mathrm{Hom}(F_n, K)/K)$, in which case the conjecture also makes sense for $n = 2$. It may be possible to prove Conjecture 4.1 for n large enough, using the fact that $\mathrm{Out}(F_n)$ is generated by torsion elements, see e.g. [BV, Zu], and an argument like the one given by Schmidt in [Sch] for strong ergodicity of the $SL(2, \mathbb{Z})$ action on $\mathbb{T}^2$. In the case when K is abelian, the conjecture is true and originally due to Rosenblatt [Ro]. When K is abelian, stronger statements in this direction, including strong ergodicity of many subgroups, follow from work of Furman and Shalom [FS]. It is tempting to hope for some duality that links Conjecture 4.1 to

Conjecture 1.1, but this hope seems naive. Any attempt to link the two conjectures must take account of the fact that Conjecture 4.1 is true when K is abelian, and the analogue of Conjecture 1.1 fails in that setting.

It is also worthwhile to compare this paper to work where relations are sought between spectral gaps in certain (or all) representations of Out(F_n) and expansion properties of various families of finite groups, see particulary [LP, GP]. In particular, it seems likely that a strong version of Conjecture 1.1 should imply Conjecture 4.1 via an argument similar to the one in [GP].

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